Resonant phenomena in slowly perturbed elliptic billiards

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Abstract

We consider an elliptic billiard whose shape slowly changes. During slow evolution of the billiard certain resonance conditions can be fulfilled. We study the phenomena of capture into a resonance and scattering on a resonance which lead to the destruction of the adiabatic invariance in the system.

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Billiard systems are important models in different fields of physics and the theory of dynamical systems [1, 2, 3, 4, 5]. Recently the billiards with varying parameters became the object of interest [6]. In the present paper we consider effects of scattering on resonances and capture into a resonance in the dynamics of a particle in an elliptic billiard with slowly varying parameters. We use a scheme of analysis of resonant phenomena in Hamiltonian systems possessing slow and fast variables [7]; this scheme is a Hamiltonian version of a general scheme [8, 9]. These methods were designed and corresponding theorems were proved for smooth systems. In the recent paper on slowly perturbed rectangular billiards [10] it was shown that the methods can be also applied adequately for billiard systems which possess discontinuities. The present paper gives another example of a billiard system possessing effects of capture into a resonance and scattering on a resonance.

Consider a particle moving in an elliptic billiard. Let r_1, r_2 be the particle distances from foci (points O_1, O_2 correspondingly), and 2c is the distance between the foci. The Hamiltonian of the system in elliptic coordinates $\xi = r_1 + r_2, \quad \eta = r_1 - r_2$ has the following form [1] :

$$H_0 = 2p_\xi^2 \frac{\xi^2 - 4c^2}{\xi^2 - \eta^2} + 2p_\eta^2 \frac{4c^2 - \eta^2}{\xi^2 - \eta^2} + U(\xi, a), \tag{1}$$

where $U(\xi, a)$ is the potential of the billiard wall:

$$U(\xi, a) = \begin{cases} \infty, & \text{if } \xi > 2a \\ 0, & \text{if } \xi < 2a \end{cases} \tag{2}$$

We investigate the problem of an elliptic billiard with slowly changing parameters $c = c(\epsilon t), \quad a = a(\epsilon t)$ which slowly rotates with angular velocity ω. Let us start with cartesian coordinates (x, y) such that the ellipse foci are located in $(c, 0)$ and $(-c, 0)$. Let (p_x, p_y) be momenta conjugated to (x, y). Changing to new variables (P_u, u, P_v, v) by means of a canonical transformation with a generating function $W = c(p_x \cosh v \cos u - p_y \sinh v \sin u)$ (see Ref. [11]) we get the Hamiltonian

$$H = \frac{P_v^2 + P_u^2}{c^2(\cosh 2v - \cos 2u)} + \frac{\epsilon c'}{c} \cdot \frac{P_u \sin 2u - P_v \sinh 2v}{\cosh 2v - \cos 2u} + \omega \cdot \frac{P_u \sinh 2v + P_v \sin 2u}{\cosh 2v - \cos 2u} + \mathcal{U}(v, a/c), \tag{3}$$

where

$$\mathcal{U}(v, a/c) = \begin{cases} \infty, & \text{if } \cosh v > a/c \\ 0, & \text{if } \cosh v < a/c \end{cases} \tag{4}$$

New coordinates relate to the elliptic coordinates by means of the following formulas:

$$\xi = 2c\cosh v, \quad \eta = 2c\cos u. \tag{5}$$

Let us introduce the variable P_t which is canonically conjugated to time t, and the variable $\tau = \epsilon t$. Now we can consider the system as the autonomous three-dimensional system with Hamiltonian: $H'(P_u, u, P_v, v, P_t, \tau) = H + P_t$. Canonically conjugated pairs are (P_u, u), (P_v, v), and $(P_t, \epsilon^{-1}\tau)$. This Hamiltonian system is conservative. Let us consider its dynamics on some level of the Hamiltonian $H' = E$. Denote $k = 2(E - P_t)$ (in the unperturbed system $k > 0$, since the Hamiltonian (3) is greater than 0). By means of simple canonical transformation we can change to Hamiltonian $H = H' - E$ and then introduce new time variable t' such that $dt/dt'{=}c^2(\cosh 2v - \cos 2u)$ [11]. Since the value of the Hamiltonian $H = H' - E$ is equal to 0, after introducing new time variable t' new Hamiltonian $\mathcal{H}$ is obtained by multiplying the Hamiltonian H by $c^2(\cosh 2v - \cos 2u)$.

So we get the following Hamiltonian:

$$\begin{aligned} \mathcal{H} \;&=\; P_v^2 - c^2k\cosh^2 v + P_u^2 + c^2k\cos^2 u + \mathcal{U}(v, a/c) + \epsilon c' c\,[P_u \sin 2u - P_v \sinh 2v] + \\ &+\; \omega c^2\,[P_u \sinh 2v + P_v \sin 2u] = \mathcal{F}_0 + \epsilon\mathcal{F}_1 \equiv 0. \end{aligned} \tag{6}$$

The Hamiltonian (6) describes the dynamics of a particle in the elliptic billiard perturbed by slow deformation and rotation. The unperturbed problem (τ=const, $\epsilon = \omega = 0$) is integrable. In addition to conserved energy there is another constant of motion $c_1 = c_u = c_v$:

$$\begin{aligned} P_v^2 - c^2k\cosh^2 v &= c_v, \\ P_u^2 + c^2k\cos^2 u &= -c_u. \end{aligned} \tag{7}$$

We introduced variables c_u, c_v which are equal to each other in the unperturbed system and slightly differ in the perturbed system. The unperturbed system can be regarded as two uncoupled oscillators whose phase portraits are presented in Fig. 1. One can introduce action-angle variables $(I_u, \phi_u, I_v, \phi_v)$ by means of canonical transformation with a generating function $S = S(I_u, I_v, u, v, P_t, \tau)$ which contains τ, P_t as parameters. In the new variables the unperturbed Hamiltonian $\mathcal{F}_0$ transforms to $\mathcal{H}_0 = \mathcal{H}_0(I_u, I_v, P_t, \tau) = \mathcal{H}_u(I_u, P_t, \tau) + \mathcal{H}_v(I_v, P_t, \tau)$. The function S has the form

$$S = \int_{v_0}^{v} \sqrt{c_v + c^2k\cosh^2 x}\,dx + \int_{u_0}^{u} \sqrt{-c_u - c^2k\cos^2 y}\,dy. \tag{8}$$

Actions of the system have the following form (see also [12, 13]):

$$\text{Case 1} \;:\; c^2k + c_u > 0.$$

$$\begin{aligned}
I_v = \frac{1}{\pi}\int_0^{\cosh^{-1}(a/c)} dv\sqrt{c_v + c^2k\cosh^2 v} &= \\
= \frac{1}{\pi}\left[\frac{c_v + kc^2}{\sqrt{kc^2}}F(\alpha,m) - \sqrt{kc^2}E(\alpha,m)\right. &+ \left. ka\sqrt{\frac{a^2-c^2}{c_v+ka^2}}\right], \\
I_u = \frac{1}{\pi}\int_{\pi/2}^{\cos^{-1} - \sqrt{\frac{-c_u}{c^2k}}} du\sqrt{-c_u - c^2k\cos^2 u} &= \frac{1}{\pi}\left[-\frac{c_u + kc^2}{\sqrt{kc^2}}\mathbf{K}(m) + \sqrt{kc^2}\mathbf{E}(m)\right], \qquad (9)\\
\text{where } m &= -\frac{c_u}{c^2k}, \quad \alpha = \arcsin\sqrt{\frac{k(a^2-c^2)}{ka^2+c_u}}.
\end{aligned}$$

$$\text{Case 2} \;:\; c^2k + c_u < 0.$$

$$\begin{aligned}
I_v = \frac{1}{\pi}\int_{\cosh^{-1}\sqrt{\frac{-c_v}{c^2k}}}^{\cosh^{-1}(a/c)} dv\sqrt{c_v + c^2k\cosh^2 v} &= \frac{1}{\pi}\left[a\sqrt{\frac{a^2k+c_v}{a^2-c^2}} - \sqrt{-c_v}E\left(\beta,\frac{1}{m}\right)\right], \\
I_u = \frac{1}{\pi}\int_{\pi/2}^{\pi} du\sqrt{-c_u - c^2k\cos^2 u} &= \frac{\sqrt{-c_u}}{\pi}\mathbf{E}(\frac{1}{m}), \qquad (10)\\
\text{where } \beta = \arcsin\sqrt{\frac{ka^2+c_u}{k(a^2-c^2)}}.
\end{aligned}$$

Frequencies of the system have the form

$$\text{case 1} \;:$$

$$\omega_v = \frac{\partial\mathcal{H}_0}{\partial I_v} = \frac{1}{\frac{\partial I_v}{\partial c_v}} = \frac{2\pi\sqrt{c^2k}}{F(\alpha,m)}, \qquad \omega_u = \frac{\partial\mathcal{H}_0}{\partial I_u} = -\frac{1}{\frac{\partial I_u}{\partial c_u}} = \frac{2\pi\sqrt{c^2k}}{\mathbf{K}(m)}.$$

$$\text{case 2} \;:$$

$$\omega_v = \frac{2\pi\sqrt{-c_v}}{F(\beta,\frac{1}{m})}, \qquad \omega_u = \frac{2\pi\sqrt{-c_u}}{\mathbf{K}\left(\frac{1}{m}\right)}.$$

Now consider the perturbed problem ($\epsilon \neq 0$), following Ref. [7]. Let us make in the system with Hamiltonian (6) the canonical transformation of the variables

$$(P_u, u, P_v, v, P_t, \tau) \to (\bar{I}_u, \bar{\phi}_u, \bar{I}_v, \bar{\phi}_v, \bar{P}_t, \bar{\tau}) \qquad (11)$$

determined by the generating function

$$S_2 = \bar{P}_t\epsilon^{-1}\tau + S(\bar{I}_u, \bar{I}_v, u, v, \bar{P}_t, \tau). \qquad (12)$$

Formulas for the transformation of the variables have the form

$$\begin{aligned} \bar{\phi}_\alpha &= \frac{\partial S}{\partial \bar{I}_\alpha}, \quad \alpha = u, v, \\ P_v &= \frac{\partial S}{\partial v}, \quad P_u = \frac{\partial S}{\partial u}, \\ P_t = \bar{P}_t + \epsilon \frac{\partial S}{\partial \tau}, &\quad \bar{\tau} = \tau + \epsilon \frac{\partial S}{\partial \bar{P}_t}. \end{aligned} \tag{13}$$

Hamiltonian (6) in the new variables is as follows:

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_0(\bar{I}_u, \bar{I}_v, \bar{P}_t, \bar{\tau}) \ + \ \epsilon \mathcal{H}_1(\bar{I}_u, \bar{I}_v, \bar{\phi}_u, \bar{\phi}_v, \bar{P}_t, \bar{\tau}) + O(\epsilon^2), \\ \mathcal{H}_1 &= \mathcal{F}_1 + \frac{\partial \mathcal{F}_0}{\partial P_t}\frac{\partial S}{\partial \tau} - \frac{\partial \mathcal{H}_0}{\partial \tau}\frac{\partial S}{\partial \bar{P}_t}. \end{aligned} \tag{14}$$

The variables $(\bar{I}_u, \bar{\phi}_u, \bar{I}_v, \bar{\phi}_v, \bar{P}_t, \bar{\tau})$ are $O(\epsilon)$- close to the variables $(I_u, \phi_u, I_v, \phi_v, P_t, \tau)$. Henceforth the bar over the new variables is omitted and the new Hamiltonian is

$$\mathcal{H} = \mathcal{H}_0(I_u, I_v, P_t, \tau) + \epsilon \mathcal{H}_1(I_u, I_v, \phi_u, \phi_v, P_t, \tau) + O(\epsilon^2). \tag{15}$$

The differential equations of the motion have the form

$$\begin{aligned} \dot{I}_\alpha = -\epsilon \frac{\partial \mathcal{H}_1}{\partial \phi_\alpha} + O(\epsilon^2), \quad \dot{\phi}_\alpha = \omega_\alpha(I_u, I_v, P_t, \tau) + \epsilon \frac{\partial \mathcal{H}_1}{\partial I_\alpha} + O(\epsilon^2), \qquad \alpha = u, v, \\ \dot{P}_t = -\epsilon \frac{\partial \mathcal{H}_0}{\partial \tau} - \epsilon^2 \frac{\partial \mathcal{H}_1}{\partial \tau} + O(\epsilon^3), \quad \dot{\tau} = \epsilon \frac{\partial \mathcal{H}_0}{\partial P_t} + \epsilon^2 \frac{\partial \mathcal{H}_1}{\partial P_t} + O(\epsilon^3). \end{aligned} \tag{16}$$

Averaging of the right hand sides of (16) over ϕ_α and discarding terms $O(\epsilon^2)$ gives an averaged system

$$\dot{I}_\alpha = 0, \quad \dot{P}_t = -\epsilon \frac{\partial \mathcal{H}_0}{\partial \tau}, \quad \dot{\tau} = \epsilon \frac{\partial \mathcal{H}_0}{\partial P_t}. \tag{17}$$

Approximation (17) is called an adiabatic approximation [9]. Trajectories of the system (17) are called adiabatic trajectories. In the adiabatic approximation $I_{u,v} = \text{const}$. The adiabatic approximation breaks down in a vicinity of a resonant surface in the phase space where the resonance condition $k_u\omega_u + k_v\omega_v = 0$ is fulfilled (k_u, k_v are integers, $k_u^2 + k_v^2 \neq 0$). Although resonant surfaces are dense in the phase space of our three-dimensional Hamiltonian system, for small ϵ only finite number of low-order resonances are important (the order of a resonance is $k = |k_u| + |k_v|$) [7].

In the exact (non-averaged) system variables I_α are approximate adiabatic invariants, i.e. they are well-conserved in a large area of phase space (far from resonant surfaces). Near a resonant surface of a given resonance the system (15) can be transformed into standard "perturbed pendulum-like system" form. Details can be found in Ref. [7].

During the motion near (k_u, k_v) - resonance the magnitude of $J = -k_v I_u + k_u I_v$ is approximately conserved. Far from resonances the magnitudes of I_u, I_v are approximately conserved.

So, a phase point in the (I_u, I_v, P_t, τ) - space moves in a following way: while it is far from low-order resonance surfaces $k_u \omega_u(I_u, P_t, \tau) + k_v \omega_v(I_v, P_t, \tau) = 0$, it moves in a vicinity of an adiabatic curve $I_{u,v}$ =const. When it approaches a resonant surface (enters a resonant zone), it leaves the adiabatic curve and can be either captured into the resonance or scattered on the resonance. In the former case it continues its motion in the vicinity of the resonant surface until become ejected from the resonance, whereas in the latter case its adiabatic invariants I_u, I_v undergo jumps $\sim \sqrt{\epsilon}$ and after crossing the resonant zone the particle continues its motion along another adiabatic curve. Similar dynamics was investigated recently in Refs. [10, 14, 15, 16]. Corresponding dynamics of the particle in the coordinate space (x, y) can be described as follows. In the adiabatic approximation each segment of the particle trajectory in a coordinate system rotating with the billiard tangents a confocal quadrics (caustics) determined by conditions $I_{u,v}$ = const. The caustics slowly evolves in accordance with evolution of the billiard parameters. In the exact system, while a phase point is far from low-order resonance surfaces, each segment of the particle trajectory tangents a confocal quadrics which is close to the caustics determined by $I_{u,v}$ = const. While captured in a (m, n) resonance, each segment of the particle trajectory tangents a confocal quadrics which is close to the quadrics determined by conditions $J = -nI_u + mI_v$ = const, $\omega_u/\omega_v = -n/m$ = const.

Consider the case of the billiard whose parameters a, c evolve periodically in time. If an adiabatic trajectory crosses a resonant surface, it crosses the surface at the same point periodically in time. A phase point which moves near this adiabatic trajectory also crosses the resonant surface repeatedly. Accumulation of changes of I_u, I_v due to multiple passages through resonances leads to the destruction of the adiabatic invariance in the system [7].

It should be noted that the billiard could be deformed in such a way that there will be no passages through resonances and the dynamics will be determined by KAM theory [1]. Let the parameters a and c be changed in such a way that a/c =const. In the system (6) one can change from variables (P_t, τ) to $(\tilde{P}_t = c^2 k, \tilde{\tau} = \tilde{\tau}(\tau) = \int_{\tau_0}^{\tau} \frac{d\tau}{c^2})$ using simple canonical transformation which does not change other variables. Then, Hamiltonian $\mathcal{H}_0$ depends only on actions $I_u, I_v, \tilde{P}_t$ and KAM theory could be applied. This means that most of the phase

space $(I_u, I_v, \tilde{P}_t)$ is filled up by invariant tori which are close to the tori $I_u, I_v, \tilde{P}_t$=const.

We performed numerical investigations of the system. The results are shown in figures 2-6. In figures 2-3 the results of numerical investigations of slowly deforming elliptic billiards are shown. Semi-axis of a billiard, d_a and d_b, were being changed periodically with time. Jumps of the adiabatic invariant I_v on a (2,1) resonance are shown in Fig. 2. A single jump of the adiabatic invariant is shown in Fig. 3. In Figures 4-6 the results of numerical investigations of slowly rotating and deforming billiards are presented. In Fig. 4 the capture of a phase point into a (2,1) resonance is shown. The captured point moves along a resonant curve until it escapes from the resonance. In Fig. 5 a phase point that were initially captured into resonance remains captured forever. Dynamics of adiabatic invariant J of that system is shown in Fig. 6. In Fig. 7 we depicted results of numerical investigation of a deforming billiard with $\omega = 0$, $a/c = \text{const}$. Coordinates (x,y) of a particle at the moments of collisions with a wall are shown. It is easy to see that dynamics is regular (compare that picture with Fig. 8, where $a/c \neq \text{const}$).

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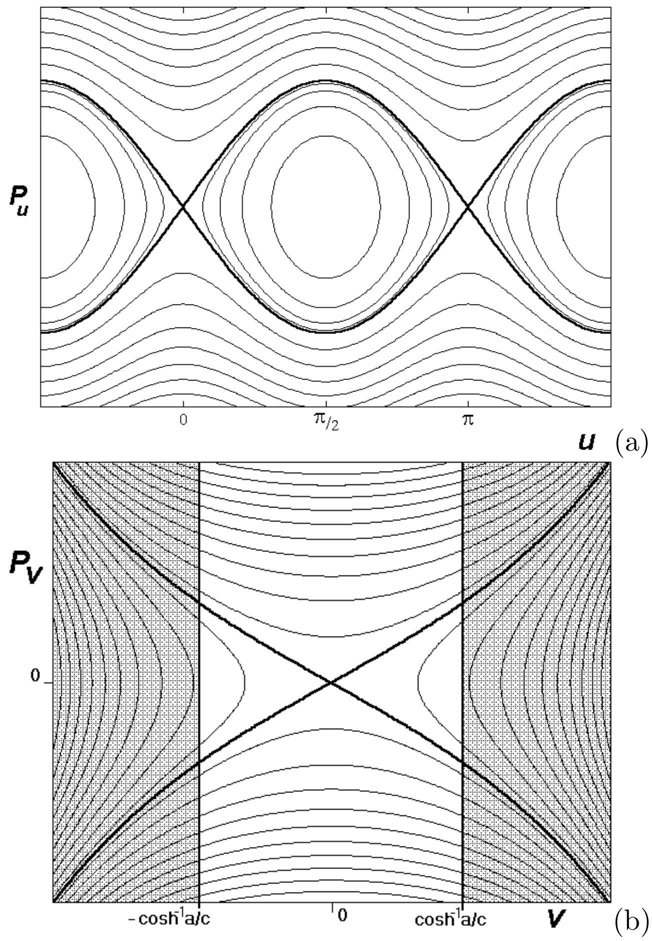


FIG. 1: Phase portraits of the oscillators (7). a) (P_u, u) plane. If $c^2k + c_u > 0$ (case 1), the motion in the (P_u, u) plane is libration. If $c^2k + c_u < 0$ (case 2), the motion is rotation. b) (P_v, v) plane. Vertical lines $v = \pm\cosh^{-1} a/c$ correspond to a potential wall of the billiard. The region between the lines corresponds to phase points moving in the billiard. Shaded region outside the lines corresponds to phase points outside the billiard (we do not consider their motion). In the case 1 ($c^2k+c_v > 0$) a phase point which begin to move from a point on the "left" wall with $P_v > 0$ goes to the right until a collision with the "right" wall, then jumps down instantaneously to the point located symmetrically about the line $P_v = 0$, then moves to the left until collision with the "left" wall, then jumps to the initial point, and so on. In the case 2 ($c^2k + c_v < 0$) a phase point which begin to move from a point on the "left" wall with $P_v > 0$ goes downwards until a collision with the same wall, then jumps to the point located symmetrically about the line $v = 0$, then moves upwards until a collision with the "right" wall, then jumps to the initial point, and so on.

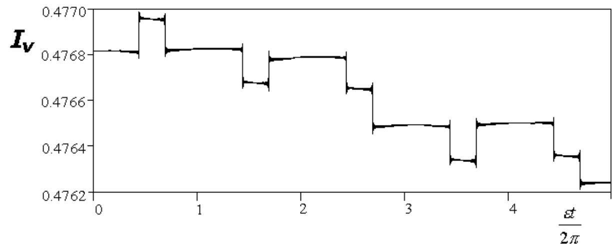


FIG. 2: Jumps of the adiabatic invariant I_v on (2,1) resonance. Parameters of the system: $\epsilon = 8{*}10^{-5}$, $\omega = 0$, d_i change harmonically with time: $d_a = d_1(1{+}A_1\cos(\epsilon t))$, $d_b = d_2(1{+}A_2\cos(\epsilon t{+}\phi))$, where $A_1 = 0.3$, $A_2 = 0.15$, $d_1 = 2$, $d_2 = 1$, $\phi = \pi/3$.

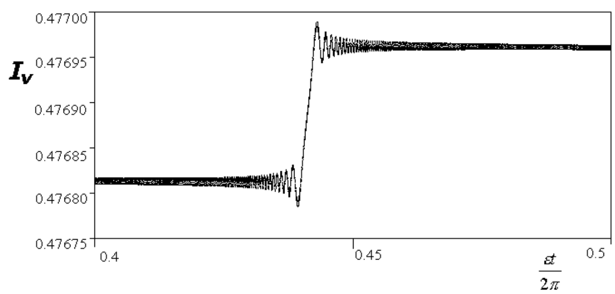


FIG. 3: A single jump of the adiabatic invariant on a (2,1) resonance. Parameters are the same as in fig 2.

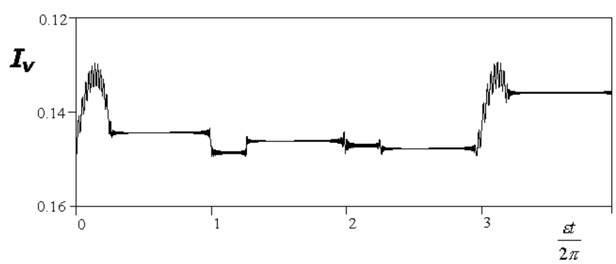


FIG. 4: Jumps of adiabatic invariant I_v and captures of a phase point into a (2,1) resonance. The captured point moves along a resonant curve until it escapes from the resonance. Parameters of the system: $\epsilon = 10^{-3}$, $\omega = 2 * 10^{-4}$, d_i change harmonically with time, as in figures 2-3 except that $A_2 = 0.2$

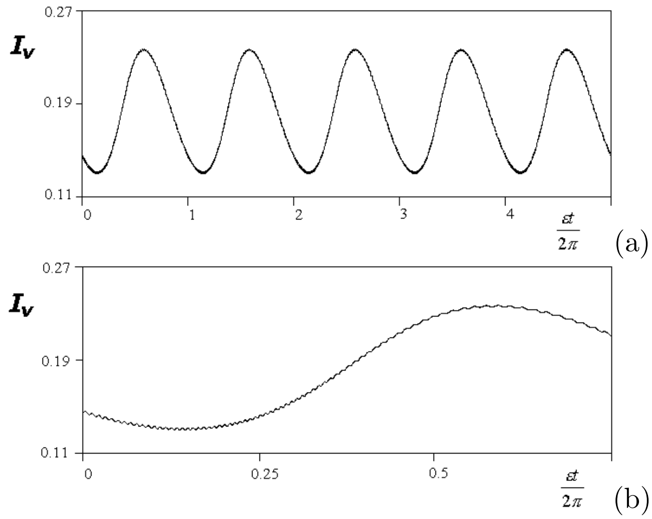


FIG. 5: Dynamics of the adiabatic invariant I_v of a phase point captured into the resonance (2,1). The phase point remains captured forever. Parameters are the same as in Fig. 4 (initial conditions are different). In Fig. a) and b) scales are different.

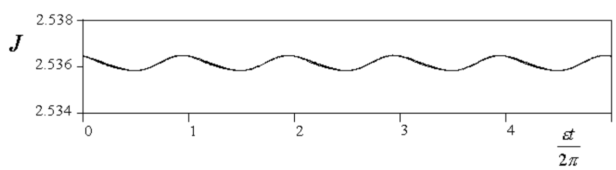
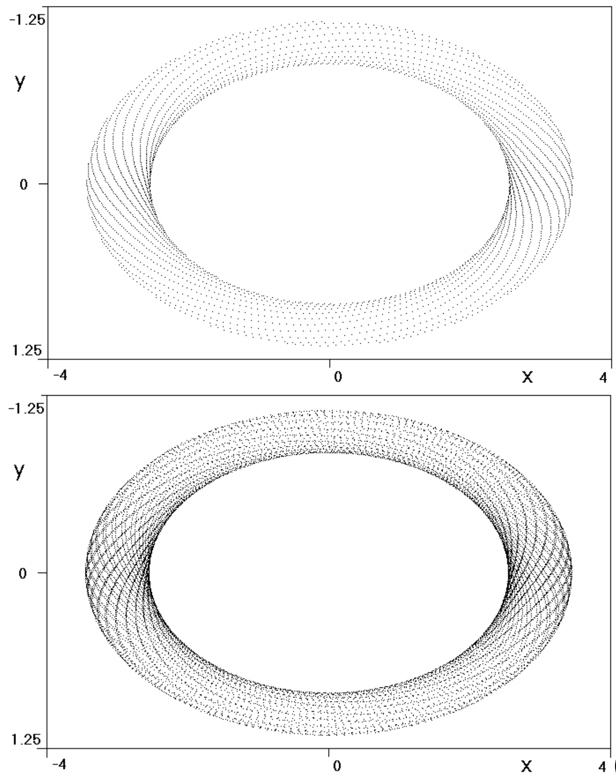


FIG. 6: Dynamics of the adiabatic invariant J. Parameters and initial conditions are the same as in Fig 5.



(a)

(b)

FIG. 7: Regular dynamics of the system with a/c =const. Coordinates (x, y) of a particle at the moments of collisions with a wall are shown. Parameters of the system: $\epsilon = 0.001$, $\omega = 0$, $A_1 = 0.15$, $A_2 = 0.15$, $d_1 = 3$, $d_2 = 1$, $\phi = 0$. a) Time of the integration is one-half of the slow period. b)Time of the integration is equal to 2 slow periods.

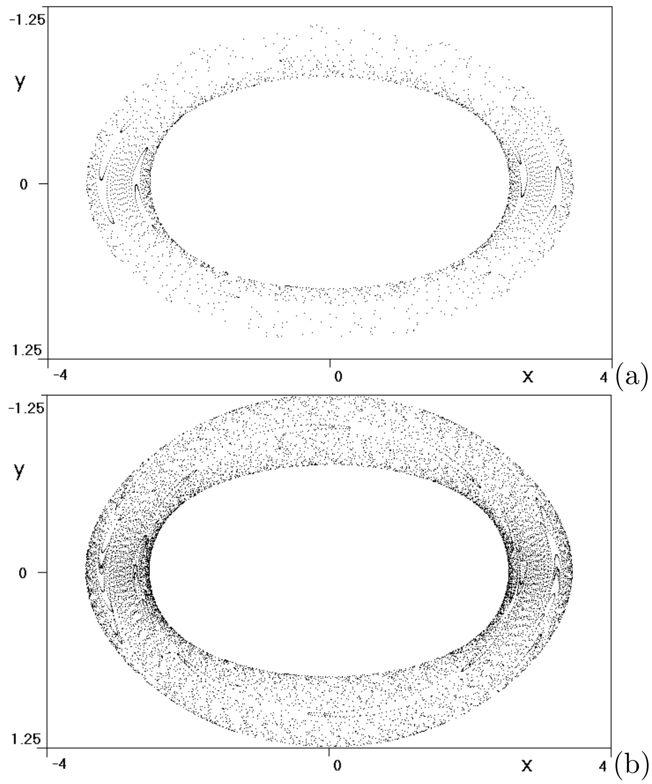


FIG. 8: Irregular dynamics of the system with $a/c \neq$const. Coordinates (x, y) of a particle at the moments of collisions with a wall are shown. Parameters of the system: $\epsilon = 0.001$, $\omega = 0$, $A_1 = 0.15$, $A_2 = 0.25$, $d_1 = 3$, $d_2 = 1$, $\phi = \pi/3$. a) Time of the integration is one-half of the slow period. b)Time of the integration is equal to 2 slow periods.